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BOUNDARY VALUE PROBLEMS FOR SECOND ORDER DIFFERENTIAL EQUATIONS IN CONVEX SUBSETS OF A BANACH SPACE

BY

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ABSTRACT. Let E be a real Banach space, C a closed, convex subset of E and f: $[0, 1] \times E \times E \longrightarrow E$ be continuous. Let $u_0, u_1 \in C$ and consider the boundary value problem

(*) $$u'' = f(t, u, u'), \quad u(0) = u_0, \quad u(1) = u_1.$$

We establish sufficient conditions in order that (*) have a solution u: $[0, 1] \longrightarrow C$.

Introduction. Let C be a closed, convex subset of the real Banach space E and let f: $[0, 1] \times C \times E \longrightarrow E$ be a function with the property

$$(1) \qquad \left.\begin{array}{l} \varphi \in E^* \text{ (2)}, x \in C, \varphi(x) = \max_{q\in C}\varphi(q) \\ y \in E, \varphi(y) = 0, 0 \leqslant t \leqslant 1 \end{array}\right\} \Rightarrow \varphi(f(t, x, y)) \geqslant 0.$$

In this paper we show that under some additional (sometimes rather restrictive) assumptions the boundary value problem (BVP)

$$(2) \qquad u'' = f(t, u, u'), \quad u(0) = u_0, \quad u(1) = u_1, \quad 0 \leqslant t \leqslant 1,$$

($u_0, u_1 \in C$) has a solution u: $[0, 1] \longrightarrow C$. We note that (1) describes the behavior of f on the boundary ∂C of C, for if $\varphi \neq 0$, then condition (1) implies $x \in \partial C$. In case $E = \mathbf{R}^n$, n-dimensional Euclidean space, and C is bounded with int C(3) $\neq \varnothing$, various results of this type exist in the literature (see e.g. [5] for a survey of such results). In this finite dimensional situation the general case may easily be obtained by projection methods. On the other hand, if E is infinite dimensional, certain additional assumptions, either on E or on f seem to be needed to pass from the case int $C \neq \varnothing$ to the general case.

The paper is divided into two parts. In the first part we assume $f(t, x, y)$

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(2) E^* denotes the space of all bounded linear functionals on E.

(3) "int" denotes the interior of a set.

to be completely continuous and satisfy a Nagumo type growth condition with respect to y. Then it is known [6] that if C is bounded and int $C \neq \emptyset$, the BVP (2) has a solution u: $[0, 1] \longrightarrow C$. In Theorem 1 we show that the same conclusion holds in case C is a closed, bounded, convex subset of a uniformly convex space E, or in case C is a compact convex subset. (The existence of a solution u: $[0, 1] \longrightarrow C$ of (2) for certain compact convex C in l^p, $1 < p < \infty$, has already been treated by Thompson [7]; his methods, however, are quite different from ours.) In the second part we assume f in (2) to be independent of u', continuous on $[0, 1] \times E$ and satisfy a Lipschitz condition

$$\|f(t, x) - f(t, y)\| \leqslant L\|x - y\|, \qquad x, y \in E, \tag{3}$$

where $L < \pi^2$. Under these assumptions the existence of a unique solution u: $[0, 1] \longrightarrow E$ of (2) follows easily by means of the contraction mapping principle, see e.g. [1] where the one dimensional case is treated, so one only needs to show that u: $[0, 1] \longrightarrow C$. This is done (Theorem 2) by using results and techniques formerly used by Redheffer and Walter [4] and in [8], [9], [10] in the study of invariance properties of sets relative to initial value problems for first order equations. A final result (Theorem 3) shows that it suffices to assume f to be defined on $[0, 1] \times C$, provided the continuity of f relative to t is uniform with respect to $x \in C$.

1. **Completely continuous right-hand sides.** Throughout this section we assume that f: $[0, 1] \times C \times E \longrightarrow E$ is completely continuous.

THEOREM 1. *Let C be a closed, bounded, convex subset of E and assume there exists a continuous projection P: $E \longrightarrow C$ assigning to each $x \in E$ a nearest point $Px \in C$ (i.e., $\|x - Px\| = \operatorname{dist}(C, x) \equiv \inf_{q \in C}\|q - x\|$; such P always exists if the Banach space E is uniformly convex in the sense of Clarkson* [2]), *or assume C is compact. Let $u_0, u_1 \in C$ and let f satisfy* (1) *and the growth condition*

$$\|f(t, x, y)\| \leqslant \omega(\|y\|) \qquad (0 \leqslant t \leqslant 1, x \in C, y \in E), \tag{4}$$

where ω: $[0, \infty) \longrightarrow (0, \infty)$ is a continuous nondecreasing function with $\lim_{s\to\infty} s^2/\omega(s) = \infty$. Then the BVP (2) *has a solution u: $[0, 1] \longrightarrow C$.*

PROOF. 1. If C is closed, bounded, convex and int $C \neq \emptyset$, the above result holds without further assumptions on C, [6, Theorem 4.1].

2. A further result [6, Lemma 2.1] which is needed in what is to follow and which makes use of the properties of ω is the following: For each $R > 0$ there exists M (depending only on R and ω) such that: if u: $[0, 1] \longrightarrow E$ is twice continuously differentiable and

$$\|u(t)\| \leqslant R, \quad \|u''(t)\| \leqslant \omega(\|u'(t)\|), \qquad 0 \leqslant t \leqslant 1,$$

then $\|u'(t)\| \leqslant M$, $0 \leqslant t \leqslant 1$.

3. Let C be such that there exists a continuous projection $P: E \longrightarrow C$ as in the statement of Theorem 1. Define $\tilde{f}: [0, 1] \times E \times E \longrightarrow E$ by

$$\tilde{f}(t, x, y) = f(t, Px, y).$$

For each $\epsilon > 0$ the set C_ϵ defined by

$$C_\epsilon = \{x \in E: \operatorname{dist}(C, x) \leqslant \epsilon\}$$

is a closed, bounded, convex subset of E with int $C_\epsilon \neq \emptyset$. We shall show next that the result of [6] stated in 1. above may be applied to $\tilde{f}$ and C_ϵ.

Obviously $\tilde{f}$ is completely continuous and verifies the estimate

$$\|\tilde{f}(t, x, y)\| \leqslant \omega(\|y\|) \qquad (0 \leqslant t \leqslant 1, x, y \in E). \tag{5}$$

Let us show (1) with C and f replaced by C_ϵ and $\tilde{f}$, respectively, i.e.

$$\left.\begin{array}{l}\varphi \in E^*, x \in C_\epsilon, \varphi(x) = \max_{q \in C_\epsilon} \varphi(q) \\ y \in E,\ \varphi(y) = 0, 0 \leqslant t \leqslant 1\end{array}\right\} \Rightarrow \varphi(\tilde{f}(t, x, y)) \geqslant 0. \tag{6}$$

Let $x \in C_\epsilon$, then $\|x - Px\| \leqslant \epsilon$. Thus, if $q \in C$, we have that $q + x - Px \in C_\epsilon$. The hypotheses of (6) consequently imply

$$\varphi(x) \geqslant \varphi(q + x - Px) = \varphi(q) + \varphi(x) - \varphi(Px),$$

and since $q \in C$ was arbitrary, it follows that

$$\varphi(Px) = \max_{q \in C} \varphi(q).$$

Using (1), we therefore obtain

$$\varphi(\tilde{f}(t, x, y)) = \varphi(f(t, Px, y)) \geqslant 0,$$

proving (6).

Using Theorem 4.1 of [6] we conclude the existence of a solution u_ϵ: $[0, 1] \longrightarrow C_\epsilon$ of the BVP

$$u_\epsilon'' = \tilde{f}(t, u_\epsilon, u_\epsilon'), \quad u_\epsilon(0) = u_0, \quad u_\epsilon(1) = u_1. \tag{7}$$

4. We now employ a limiting process (letting $\epsilon \longrightarrow 0$) to obtain the desired conclusion.

Let $\{\epsilon_n\}$ be a monotone decreasing sequence of real numbers with $\lim_{n\to\infty} \epsilon_n = 0$. Denote by $u_n = u_{\epsilon_n}$, where u_{ϵ_n}: $[0, 1] \longrightarrow C_{\epsilon_n}$ is a solution of (7), with ϵ replaced by ϵ_n. Choose $R > 0$ such that $\|u_n(t)\| \leqslant R$, $0 \leqslant t \leqslant 1$, $n = 1, 2, \ldots$. Using (5) and 2. we obtain the existence of a constant $M > 0$ such that $\|u_n'(t)\| \leqslant M$, $0 \leqslant t \leqslant 1$, $n = 1, 2, \ldots$.

Let G denote the Green's function

$$G(t, s) = \begin{cases} -s(1-t), & 0 \leq s \leq t \leq 1, \\ -t(1-s), & 0 \leq t \leq s \leq 1; \end{cases}$$

then

$$u_n(t) = \int_0^1 G(t, s)\tilde{f}(s, u_n(s), u_n'(s))\, ds + (1-t)u_0 + tu_1 \tag{8}$$

and

$$u_n'(t) = \int_0^1 \frac{\partial}{\partial t} G(t, s)\tilde{f}(s, u_n(s), u_n'(s))\, ds + u_1 - u_0. \tag{9}$$

Using the complete continuity of $\tilde{f}$, the uniform boundedness of $\{u_n\}$, $\{u_n'\}$ and (8), (9) we conclude that $\{u_n\}$ and $\{u_n'\}$ are equicontinuous sequences and that there exists a compact set $K \subseteq E$ such that $u_n(t), u_n'(t) \in K$, $0 \leq t \leq 1$, $n = 1, 2, \ldots$.

We may thus employ the theorem of Ascoli-Arzelà to obtain a subsequence of $\{u_n\}$ which converges to a solution u of

$$u'' = \tilde{f}(t, u, u'), \quad u(0) = u_0, \quad u(1) = u_1, \qquad 0 \leq t \leq 1.$$

Since, further, $\mathrm{dist}(C, u_n(t)) \leq \epsilon_n$ and $\lim_{n\to\infty} \epsilon_n = 0$, we obtain $\mathrm{dist}(C, u(t)) = 0$, from which follows that u: $[0, 1] \longrightarrow C$ and $\tilde{f}(t, u, u') = f(t, u, u')$, proving that u is a solution of (2).

5. We next consider the case where C is a compact convex subset of E (here no additional assumptions on E are needed). Choose $R > 0$ such that: $x \in C \Rightarrow \|x\| \leq R$. Determine $M = M(R, \omega)$ according to 2. above. Define Q: $E \longrightarrow E$ by

$$Qy = \begin{cases} y, & \|y\| \leq M, \\ My/\|y\|, & \|y\| > M, \end{cases}$$

and put

$$\tilde{f}(t, x, y) = f(t, x, Qy) \qquad (0 \leq t \leq 1, x \in C, y \in E).$$

The complete continuity of f implies that of $\tilde{f}$. Hence the range of $\tilde{f}$ is contained in some compact set $K \subseteq E$, and (1) and (4) are satisfied by $\tilde{f}$.

Let E_1 denote the closed linear span of C, K and restrict $\tilde{f}$ to $\tilde{f}$: $[0, 1] \times C \times E_1 \longrightarrow E_1$. Since C and K are compact, E_1 is a separable Banach space. Using a result of Clarkson [2] we may equip E_1 with a new norm $\|\cdot\|_1$, equivalent to $\|\cdot\|$, such that E_1 becomes strictly convex. Hence to each $x \in E_1$ there corresponds a unique nearest point (with respect to $\|\cdot\|_1$) Px in C. Since (1) holds with E, f replaced by E_1, $\tilde{f}$ ($\varphi \in E_1^*$ with $\varphi(x) = \max_{q\in C}\varphi(q)$ is extendable to a $\Phi \in E^*$ with the same property) and since $\tilde{f}$ is bounded and the projection P, just defined, is continuous, we may apply the arguments of 3. and 4. to obtain a solution u: $[0, 1] \longrightarrow C$ of

$$u'' = \widetilde{f}(t, u, u'), \quad u(0) = u_0, \quad u(1) = u_1, \qquad 0 \leqslant t \leqslant 1. \tag{10}$$

Returning to the original norm we have that $\|u(t)\| \leqslant R$, $0 \leqslant t \leqslant 1$, and by the monotonicity of ω we find $\|u''(t)\| \leqslant \omega(\|u'(t)\|)$, implying $\|u'(t)\| \leqslant M$, $0 \leqslant t \leqslant 1$. Hence the definition of $\widetilde{f}$ shows that u is a solution of (2).

2. **Right-hand sides satisfying a Lipschitz condition.** Throughout this section we shall assume that f is independent of u' and satisfies a Lipschitz condition

$$\|f(t, x) - f(t, y)\| \leqslant L\|x - y\| \qquad (0 \leqslant t \leqslant 1; x, y \in E). \tag{11}$$

THEOREM 2. *Let C be a closed, convex subset of E and let $u_0, u_1 \in C$. Assume that f: $[0, 1] \times E \longrightarrow E$ is continuous and satisfies the Lipschitz condition* (11) *with $L < \pi^2$. Further assume*

$$\left(0 \leqslant t \leqslant 1, \varphi \in E^*, x \in C, \varphi(x) = \max_{q \in C} \varphi(q)\right) \Rightarrow \varphi(f(t, x)) \geqslant 0. \tag{12}$$

Then the BVP

$$u'' = f(t, u), \quad u(0) = u_0, \quad u(1) = u_1, \qquad 0 \leqslant t \leqslant 1, \tag{13}$$

has a unique solution u: $[0, 1] \longrightarrow C$.

PROOF. 1. For our proof we need a formula first established for closed, convex cones by Redheffer and Walter [4] equivalent to (12):

$$\lim_{h \to 0+} \frac{1}{h} \operatorname{dist}(C, x - hf(t, x)) = 0 \qquad (0 \leqslant t \leqslant 1, x \in C) \tag{14}$$

(see [8]). Letting (for $\xi \geqslant 0$)

$$C_\xi = \{x \in E: \operatorname{dist}(C, x) \leqslant \xi\}$$

($C_0 = C$), using (11) and (12) and a result from [10] we obtain

$$\lim_{h \to 0+} \frac{1}{h} \operatorname{dist}(C_\xi, x - hf(t, x)) \leqslant L\xi \qquad (0 \leqslant t \leqslant 1, x \in C_\xi). \tag{15}$$

(In [10] this formula is written with lim sup in place of lim, however, since C is convex the limit exists.)

2. Let $\widetilde{E} = E \oplus \mathbf{R}$ normed by $\|(x, \xi)\| = \max(\|x\|, |\xi|)$. With $p = (\theta, 1)$ (θ = zero element of E) we may write

$$\widetilde{E} = E \oplus \mathbf{R} = \{x + \xi p: x \in E, \xi \in \mathbf{R}\}.$$

Via the natural embedding, we consider E as a subspace of $\widetilde{E}$. Let

$$\widetilde{C} = \{x + \xi p: \operatorname{dist}(C, x) \leqslant \xi\},$$

then $\widetilde{C}$ is a closed, convex subset of $\widetilde{E}$ with nonempty interior. Define $\widetilde{f}$: $[0, 1]$

$\times \widetilde{C} \longrightarrow \widetilde{E}$ by

$$\widetilde{f}(t, x + \xi p) = f(t, x) - L\xi p \qquad (0 \leqslant t \leqslant 1, x + \xi p \in \widetilde{C}). \tag{16}$$

Then $\widetilde{f}$ is continuous and satisfies a Lipschitz condition with Lipschitz constant L with respect to its second argument:

$$\|\widetilde{f}(t, \widetilde{x}) - \widetilde{f}(t, \widetilde{y})\| \leqslant L\|\widetilde{x} - \widetilde{y}\| \qquad (0 \leqslant t \leqslant 1, \widetilde{x}, \widetilde{y} \in \widetilde{C}). \tag{17}$$

Our method of proof requires a condition analogous to (12) for $\widetilde{f}$ and $\widetilde{C}$, namely:

$$\left(0 \leqslant t \leqslant 1, \widetilde{\varphi} \in \widetilde{E}^*, \widetilde{x} \in \widetilde{C}, \widetilde{\varphi}(\widetilde{x}) = \max_{\widetilde{q} \in \widetilde{C}} \widetilde{\varphi}(\widetilde{q})\right) \Rightarrow \widetilde{\varphi}(\widetilde{f}(t, \widetilde{x})) \geqslant 0. \tag{18}$$

That (18) follows from (12) has already been sketched in [9] for the case where C is a closed, convex cone; our proof to follow is patterned after the one in [9]. (For general closed, convex C (18) has been established in [8] for f defined by $\widetilde{f}(t, x + \xi p) = f(t, x) - 4L\xi p$. That result, however, is not sufficient for our purposes.)

3. To prove (18) we use the equivalence of (12) and (14) (applied to $\widetilde{C}$ and $\widetilde{f}$) and verify

$$\lim_{h \to 0+} \frac{1}{h} \operatorname{dist}(\widetilde{C}, \widetilde{x} - h\widetilde{f}(t, \widetilde{x})) = 0 \qquad (0 \leqslant t \leqslant 1, \widetilde{x} \in \widetilde{C}). \tag{19}$$

Let $t \in [0, 1]$ and $\widetilde{x} = x + \xi p \in \widetilde{C}$, i.e., $x \in C_\xi$. Then (15) implies that for $\epsilon > 0$ there exists $h_0(\epsilon)$ such that

$$h^{-1} \operatorname{dist}(C_\xi, x - hf(t, x)) < L\xi + \epsilon \qquad (0 < h \leqslant h_0(\epsilon)).$$

Thus there exists $y_h \in C_\xi$ (i.e. $y_h + \xi p \in \widetilde{C}$) such that

$$\|x - hf(t, x) - y_h\| < hL\xi + h\epsilon,$$

implying

$$x - hf(t, x) - y_h + h(L\xi + \epsilon)p \in \widetilde{K} \equiv \{y + \eta p \colon y \in E, \|y\| \leqslant \eta\}.$$

Now $\widetilde{C} + \widetilde{K} \subseteq \widetilde{C}$ and $y_h + \xi p \in \widetilde{C}$, yielding

$$x + \xi p - h[f(t, x) - L\xi p] + h\epsilon p \in \widetilde{C},$$

from which, in turn, it follows that

$$h^{-1} \operatorname{dist}(\widetilde{C}, \widetilde{x} - h\widetilde{f}(t, \widetilde{x})) \leqslant \epsilon \qquad (0 < h \leqslant h_0(\epsilon)),$$

implying (19).

4. Define $P \colon \widetilde{E} \longrightarrow \widetilde{C}$ by

$$P(x + \xi p) = \begin{cases} x + \xi p, & \operatorname{dist}(C, x) \leqslant \xi, \\ x + \operatorname{dist}(C, x)p, & \operatorname{dist}(C, x) > \xi. \end{cases} \tag{20}$$

Then it is easily seen that

$$\|P(\tilde{x}) - P(\tilde{y})\| \leq \|\tilde{x} - \tilde{y}\|. \tag{21}$$

Extending $\tilde{f}$ to $[0, 1] \times \tilde{E}$ by setting

$$\tilde{f}(t, \tilde{x}) = \tilde{f}(t, P\tilde{x}) \quad (0 \leq t \leq 1, \tilde{x} \in \tilde{E}), \tag{22}$$

we see by (21) that (17) remains valid for the extended function (with the same Lipschitz constant).

Letting

$$\tilde{C}_\eta = \tilde{C} - \eta p = \{\tilde{x} - \eta p : \tilde{x} \in \tilde{C}\} \quad (\eta \geq 0; \tilde{C}_0 = \tilde{C})$$

we see that (18) holds with $\tilde{C}$ replaced by $\tilde{C}_\eta$, i.e.,

$$\left(0 \leq t \leq 1, \tilde{\varphi} \in \tilde{E}^*, \tilde{x} \in \tilde{C}_\eta, \tilde{\varphi}(\tilde{x}) = \max_{\tilde{q} \in \tilde{C}_\eta} \tilde{\varphi}(\tilde{q})\right) \Rightarrow \tilde{\varphi}(\tilde{f}(t, \tilde{x})) \geq 0, \tag{23}$$

for if $\tilde{x} = x + \xi p$ and $\tilde{\varphi} \neq 0$ satisfy the hypotheses of (23), then $\tilde{x} \in \partial \tilde{C}_\eta$ and therefore $x + (\xi + \eta)p = \tilde{x} + \eta p \in \partial \tilde{C}$. Thus $\operatorname{dist}(C, x) = \xi + \eta$, which combined with (20) yields $P\tilde{x} = x + (\xi + \eta)p = \tilde{x} + \eta p$. Therefore $\tilde{\varphi}(P\tilde{x}) = \max_{\tilde{q} \in \tilde{C}} \tilde{\varphi}(\tilde{q})$. Using (18) we obtain $\tilde{\varphi}(\tilde{f}(t, P\tilde{x})) \geq 0$, which by (22) implies (23).

5. The function $\sigma: \tilde{E} \to \mathbf{R}$, defined by

$$\sigma(x + \xi p) = \begin{cases} 0, & \operatorname{dist}(C, x) \leq \xi, \\ \operatorname{dist}(C, x) - \xi, & \operatorname{dist}(C, x) > \xi, \end{cases} \tag{24}$$

satisfies a Lipschitz condition with Lipschitz constant 2. Choose $\epsilon > 0$ such that $L_1 = L + 2\epsilon < \pi^2$. Then

$$\hat{f}(t, \tilde{x}) = \tilde{f}(t, \tilde{x}) - \epsilon\sigma(\tilde{x})p$$

satisfies

$$\|\hat{f}(t, \tilde{x}) - \hat{f}(t, \tilde{y})\| \leq L_1 \|\tilde{x} - \tilde{y}\| \quad (0 \leq t \leq 1, \tilde{x}, \tilde{y} \in \tilde{E});$$

further it follows from (23) and (24) that

$$\left(0 \leq t \leq 1, \eta > 0, \tilde{\varphi} \in \tilde{E}^*, \tilde{\varphi} \neq 0, \tilde{x} \in \tilde{C}_\eta, \tilde{\varphi}(\tilde{x}) = \max_{\tilde{q} \in \tilde{C}_\eta} \tilde{\varphi}(\tilde{q})\right) \Rightarrow \tilde{\varphi}(\hat{f}(t, \tilde{x})) > 0. \tag{25}$$

Because $L_1 < \pi^2$, the BVP

$$\tilde{u}'' = \hat{f}(t, \tilde{u}), \quad \tilde{u}(0) = u_0, \quad \tilde{u}(1) = u_1, \tag{26}$$

has a unique solution $\tilde{u}: [0, 1] \to \tilde{E}$ (this fact has already been mentioned in the introduction). It is the purpose of the next paragraphs to show that $\tilde{u}$ is a solution of (13) with values in C.

6. There exists a smallest $\eta \geq 0$ such that $\widetilde{u}$: $[0, 1] \longrightarrow \widetilde{C}_\eta$ ($\widetilde{u}$ is the solution of (26)). Suppose $\eta > 0$. Then there exists $t_0 \in (0, 1)$ such that $\widetilde{u}(t_0) \in \partial\widetilde{C}_\eta$ ($\widetilde{u}(0), \widetilde{u}(1) \in \operatorname{int} \widetilde{C}_\eta$). We may thus choose $\widetilde{\varphi} \in \widetilde{E}^*$, $\widetilde{\varphi} \neq 0$, such that $\widetilde{\varphi}(\widetilde{u}(t_0)) = \max_{\widetilde{q} \in \widetilde{C}_\eta} \widetilde{\varphi}(\widetilde{q})$. By (25)

$$\widetilde{\varphi}(\hat{f}(t_0, \widetilde{u}(t_0))) > 0. \tag{27}$$

On the other hand, the scalar function $\rho(t) = \widetilde{\varphi}(\widetilde{u}(t))$, $0 \leq t \leq 1$, attains its maximum at t_0, hence $\rho''(t_0) \leq 0$. But

$$\rho''(t_0) = \widetilde{\varphi}(\widetilde{u}''(t_0)) = \widetilde{\varphi}(\hat{f}(t_0, \widetilde{u}(t_0))),$$

contradicting (27). Thus $\widetilde{u}$: $[0, 1] \longrightarrow \widetilde{C}_0 = \widetilde{C}$.

7. It now follows from the definition of $\hat{f}$ that $\hat{f}(t, \widetilde{u}(t)) = \widetilde{f}(t, \widetilde{u}(t))$. Thus $\widetilde{u}$ is the solution of the BVP

$$\widetilde{u}'' = \widetilde{f}(t, \widetilde{u}), \; \widetilde{u}(0) = u_0, \quad \widetilde{u}(1) = u_1. \tag{28}$$

Using the notation

$$\widetilde{u}(t) = u(t) + \xi(t)p \qquad (u(t) \in E,\ \xi(t) \in \mathbf{R},\ 0 \leq t \leq 1),$$

we may decompose (28) into

$$u'' = f(t, u), \quad u(0) = u_0, \quad u(1) = u_1, \tag{29}$$

$$\xi'' = -L\xi, \qquad \xi(0) = 0, \qquad \xi(1) = 0, \tag{30}$$

$$0 \leq t \leq 1,$$

with the further constraint

$$\operatorname{dist}(C, u(t)) \leq \xi(t). \tag{31}$$

Since, however, $L < \pi^2$, it follows that $\xi(t) \equiv 0$, and thus $\operatorname{dist}(C, u(t)) = 0$, i.e., u: $[0, 1] \longrightarrow C$. This completes the proof of Theorem 2.

THEOREM 3. *Theorem 2 remains valid if $f(t, x)$ is only defined on* $[0, 1] \times C$, *but is uniformly continuous in t with respect to x, i.e.,*

$$\sup_{x \in C} \|f(t_n, x) - f(t, x)\| \longrightarrow 0 \quad \text{as } t_n \longrightarrow t. \tag{32}$$

PROOF. We embed E via an isometric isomorphism in some Banach space $B(S)$ of bounded functions on some set S (e.g. $S = \{\varphi \in E^*: \|\varphi\| \leq 1\}$). Then (12) remains valid with $B(S)^*$ in place of E^*. Thus we may consider the problem in $B(S)$ instead of E; in particular we may consider f: $[0, 1] \times C \longrightarrow B(S)$, where $C \subseteq B(S)$. By adopting the coordinate conventions and writing the elements $z \in B(S)$ as $z = (z_\sigma)_{\sigma \in S}$ ($z_\sigma \in \mathbf{R}$, $\|z\| = \sup_{\sigma \in S} |z_\sigma|$), we define f_σ: $[0, 1] \times C \longrightarrow \mathbf{R}$ ($\sigma \in S$) by

$$f_\sigma(t, x) = f(t, x)_\sigma \qquad (0 \leq t \leq 1, x \in C, \sigma \in S).$$

The Lipschitz continuity of f implies that of f_σ, i.e.,

$$|f_\sigma(t, x) - f_\sigma(t, y)| \leqslant L\|x - y\| \qquad (0 \leqslant t \leqslant 1, x, y \in C, \sigma \in S).$$

A result of McShane [3] implies that the function

$$\tilde{f}_\sigma(t, x) = \sup_{q \in C} (f_\sigma(t, q) - L\|q - x\|) \qquad (x \in B(S))$$

is an extension of f_σ to $[0, 1] \times B(S)$, such that

$$|\tilde{f}_\sigma(t, x) - \tilde{f}_\sigma(t, y)| \leqslant L\|x - y\| \qquad (0 \leqslant t \leqslant 1, x, y \in B(S), \sigma \in S).$$

Define $\tilde{f}$: $[0, 1] \times B(S) \longrightarrow B(S)$ by

$$\tilde{f}(t, x)_\sigma = \tilde{f}_\sigma(t, x) \qquad (0 \leqslant t \leqslant 1, x \in B(S), \sigma \in S).$$

Then $\tilde{f}$ is an extension of f to $[0, 1] \times B(S)$ and satisfies (11). By (32) $\tilde{f}(t, x)$ is also continuous with respect to t. We may therefore use Theorem 2 to conclude that the BVP

$$u'' = \tilde{f}(t, u), \quad u(0) = u_0, \quad u(1) = u_1 \qquad (u_0, u_1 \in C)$$

has a solution u: $[0, 1] \longrightarrow C$. Since $\tilde{f}$ is an extension of f, u is a solution of the original problem.

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